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TECHNICAL NOTES

Non-Darcy natural convection over a slender vertical frustum of a cone in a saturated porous medium

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INTRODUCTION

IN RECENT years intensive interest has been shown in natural convection in porous media of various configurations including vertical and horizontal layers, sloped layers etc. (see Cheng [1, 2] for a review of the literature). Such interest may be explained by the fact that porous media have numerous applications in geophysics and energy-related engineering problems. In almost all published papers, boundary-layer formulation of Darcy's law and the energy equation were used. However, it was shown by Bear [3] that deviations from Darcy's law occur when the Reynolds number based on the averaged grain diameter is smaller than about 10. Recently, Plumb and Huenefeld [4] made a theoretical investigation of the natural convection boundary layers adjacent to a vertical heated surface immersed in a saturated porous medium using a non-Darcy flow model proposed by Ergun [5]. This paper has been followed by other experimental and theoretical studies performed by Cheng *et al.* [6] and Huenefeld and Plumb [7].

In the present paper the steady non-Darcy natural convection from an isothermal slender vertical frustum of a cone embedded in a saturated porous medium is considered using the Ergun model. We have included the transverse curvature terms in the governing differential equations, that is, the boundary-layer thickness is assumed to be of the same order of magnitude as the local radius of the cone or that the cone angle is small. Analogous to the analyses of the natural convection boundary-layer flow over a frustum of a cone in a viscous fluid by Na and Chiou [8, 9] the flow is non-similar. Numerical solutions are therefore obtained of the governing differential equations using Keller's box method [10], a method that has been shown to be particularly accurate for parabolic problems. Referring to Fig. 1, the flow over a frustum of a cone will approach to the flow over a full cone as x_0 approaches to zero. The results have been compared with those of Plumb and Huenefeld [4].

GOVERNING EQUATIONS

Consider non-Darcy natural convection over a slender vertical frustum of a cone immersed in a fluid-saturated porous medium with constant fluid and medium (isotropic) properties and local thermodynamic equilibrium between fluid and solid phases. The physical model and the coordinate system are shown in Fig. 1. The boundary layer is assumed to develop at the leading edge ($x = x_0$), which means the temperature at the circular base is the same as the temperature of the surrounding fluid. Due to the difference in temperature between the surface and the surrounding fluid, an upward flow is created as a result of buoyancy forces. If the Boussinesq approximation is employed it can be shown that the governing equations with boundary-layer simplifications can be written in terms of dimensionless variables as

$$\frac{\partial}{\partial \hat{x}}(\hat{r}\hat{u})+\frac{\partial}{\partial \hat{y}}(\hat{r}\hat{v}) = 0 \tag{1}$$

$$\frac{\partial \hat{u}}{\partial \hat{y}}+Gr^*\frac{\partial}{\partial \hat{y}}(\hat{u}^2) = \frac{\partial \theta}{\partial \hat{y}} \tag{2}$$

NOMENCLATURE

f	reduced stream function
g	acceleration due to gravity
Gr^*	modified Grashof number
h	local heat transfer coefficient
k_m	thermal conductivity of the porous medium
K	permeability
K^*	inertial coefficient
Nu_{x^*}	local Nusselt number
r_0	radius of the cone
r	radial distance from the axis of the cone
Ra_{x_0}	modified Rayleigh number based on x_0
Ra_{x^*}	modified Rayleigh number based on x
T	temperature
u_r	reference velocity
u, v	velocity components in x and y directions
x_0	location along the surface of the cone
x, y	rectangular coordinates.

Greek symbols

α	equivalent thermal diffusivity
β	coefficient of thermal expansion
γ	transverse curvature parameter
ξ, η	transformed coordinates
θ	dimensionless temperature function
ν	kinematic viscosity
ρ	fluid density
Φ	cone angle
ψ	stream function.

Superscript

$'$	prime denotes differentiation with respect to η.

Subscripts

w	wall condition
∞	surroundings condition
TVC	transverse curvature effects
NO-TVC	no transverse curvature effects.

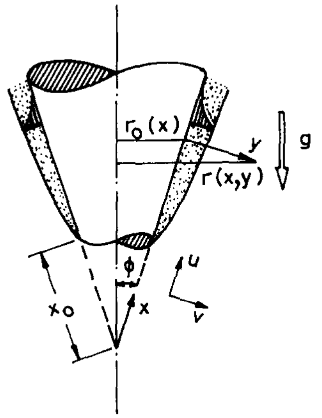

FIG. 1. Physical model and coordinate system.

$$\hat{u}\frac{\partial \theta}{\partial \hat{x}}+\hat{v}\frac{\partial \theta}{\partial \hat{y}}=\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{y}}\left(\hat{r}\frac{\partial \theta}{\partial \hat{y}}\right) \tag{3}$$

subject to the boundary conditions

$$\begin{aligned} &\hat{v}=0, \quad \theta=1 \quad \text{at } \hat{y}=0 \\ &\hat{u}\to 0, \quad \theta \to 0 \quad \text{as } y\to\infty \end{aligned} \tag{4}$$

and where

$$Gr^* = g\beta KK^*(T_w - T_\infty)\cos\Phi/v^2$$

is a modified Grashof number expressing the relative importance of the inertial effects. The non-dimensional variables employed are

$$\hat{x}=(x-x_0)/x_0, \quad \hat{y}=(y/x_0)Ra_{x_0}^{1/2},$$
$$\hat{u}=u/u_r, \quad \hat{v}=(v/u_r)Ra_{x_0}^{1/2}$$
$$\hat{r}=r(x,y)/x_0, \quad \hat{r}_0=r_0(x)/x_0,$$
$$\theta=(T-T_\infty)/(T_w-T_\infty)$$

in which

$$u_r = g\beta K(T_w-T_\infty)\cos\Phi/v, \quad Ra_{x_0}=u_r x_0/\alpha$$

Ra_{x_0} being the modified Rayleigh number for a porous medium based on the length x_0.

Next, the coordinates (ξ,η) and the reduced stream function f are defined as follows

$$\xi=\hat{x}, \quad \eta=\hat{y}/\hat{x}^{1/2}, \quad f(\xi,\eta)=\psi\hat{x}^{-1/2}/\hat{r}_0 \tag{5}$$

where the dimensionless stream function ψ is introduced in the usual way

$$\hat{r}\hat{u}=\partial\psi/\partial\hat{y}, \quad \hat{r}\hat{v}=-\partial\psi/\partial\hat{x} \tag{6}$$

and, for cones

$$r_0 = x\sin\Phi \quad \text{or} \quad \hat{r}_0=(1+\hat{x})\sin\Phi. \tag{7}$$

When the transformations (5)–(7) are applied to equations (2) and (3), there results

$$[(\hat{r}_0/\hat{r})f']' + Gr^*\{[(\hat{r}_0/\hat{r})f']^2\}' = \theta' \tag{8}$$

$$[(\hat{r}_0/\hat{r})\theta']'+(2^{-1}+S)f\theta' = \xi[f'(\partial\theta/\partial\xi)-\theta'(\partial f/\partial\xi)] \tag{9}$$

with the boundary conditions

$$f(\xi,0)=0, \quad \theta(\xi,0)=1, \quad f'(\xi,\infty)=0, \quad \theta(\xi,\infty)=0 \tag{10}$$

where

$$S=(\hat{x}/\hat{r}_0)(d\hat{r}_0/d\hat{x})=\hat{x}(1+\hat{x})^{-1}=\xi(1+\xi)^{-1}. \tag{11}$$

We note that for cones, we have

$$r_0 = x\sin\Phi, \quad r=r_0+y\cos\Phi. \tag{12}$$

With r_0 and r thus defined the quantity $\hat{r}/\hat{r}_0$ appearing in (8) and (9), in terms of the similarity variables defined in equations (5) becomes

$$\hat{r}/\hat{r}_0 = 1+\gamma\xi^{1/2}(1+\xi)^{-1} \tag{13}$$

where $\gamma=\cot\Phi\ Ra_{x_0}^{-1/2}$ is a transverse curvature (TVC) parameter. Making use of (11), equations (8) and (9), after a little algebra, can be written in the form

$$f''-\gamma^*(\hat{r}_0/\hat{r})f'+2Gr^*(\hat{r}_0/\hat{r})[f'f''-\gamma^*(\hat{r}_0/\hat{r})(f')^2]=(\hat{r}/\hat{r}_0)\theta' \tag{14}$$

$$(\hat{r}/\hat{r}_0)\theta''+\gamma^*\theta'+(S+2^{-1})f\theta'=\xi[f'(\partial\theta/\partial\xi)-\theta'(\partial f/\partial\xi)] \tag{15}$$

with the boundary conditions

$$f(\xi,0)=0, \quad \theta(\xi,0)=1, \quad f'(\xi,\infty)=\theta(\xi,\infty)=0 \tag{16}$$

where

$$\gamma^*=\xi^{1/2}(1+\xi)^{-1}\gamma. \tag{17}$$

It may be noted that the ratio $\hat{r}/\hat{r}_0$ represents the effects of transverse curvature. For flows far downstream or for large cone angle Φ, γ is very close to r_0 and the effect of transverse curvature is negligible, i.e. $\gamma=0$. Under this condition equations (14) and (15), in the absence of transverse curvature effect, reduce to

$$f''+2Gr^*f'f''=\theta' \tag{18}$$

$$\theta''+(S+2^{-1})f\theta'=\xi[f'(\partial\theta/\partial\xi)-\theta'(\partial f/\partial\xi)] \tag{19}$$

with boundary conditions given by (16). It should be remarked that equations (18) and (19) are also non-similar. Also, the parameter S approaches 1 when $\hat{x}(\xi)$ becomes very large and the solution of (14) and (15) is expected to approach to the similarity solution of the non-Darcy natural convection flow over a full cone embedded in a saturated porous medium, namely

$$f''+2Gr^*f'f''=\theta' \tag{20}$$

$$\theta''+(3/2)f\theta'=0 \tag{21}$$

with the boundary conditions

$$f(0)=0, \quad \theta(0)=1, \quad f'(\infty)=\theta(\infty)=0. \tag{22}$$

The heat transfer coefficient in terms of the Nusselt number can be expressed as

$$Ra_{x^*}^{-1/2}(Nu_{x^*})_{\mathrm{TVC}}=[-\theta'(\xi,0)]_{\mathrm{TVC}} \tag{23}$$

where

$$Nu_{x^*}=hx^*/x_0, \quad h=-k_m(\partial T/\partial y)_{y=0}/(T_w-T_\infty) \tag{24}$$

and $x^*=x-x_0$. If the effects of transverse curvature are not included in (14) and (15) [i.e. equations (18) and (19) are solved instead], a similar expression can be written as

$$Ra_{x^*}^{-1/2}(Nu_{x^*})_{\text{NO-TVC}}=[-\theta'(\xi,0)]_{\text{NO-TVC}}. \tag{25}$$

RESULTS AND DISCUSSION

Equations (14)–(22) have been solved by a finite-difference scheme developed by Keller [10]. The numerical integration is started at $\xi=0$, where f, f', θ and θ' can be found from equations (18) and (19), and then proceeds in a stepwise manner. The detailed description of the method is presented in [10, 11]. Hence, for the sake of brevity, it is not presented here.

Table 1. Heat and velocity parameters for $\xi = \eta = \gamma = 0$

	Present results		Plumb and Huenefeld [4]	
Gr^*	$\theta'(0)$	$f''(0)$	$\theta'(0)$	$f''(0)$
0.0	−0.44456	1.00000	−0.44390	1.00000
0.01	−0.44292	0.99020	−0.44232	0.99020
0.1	−0.43042	0.91608	−0.42969	0.91608
1.0	−0.36662	0.61803	−0.36617	0.61803
10.0	−0.25140	0.27016	−0.25126	0.27016
100.0	−0.15212	0.09512	−0.15186	0.09512

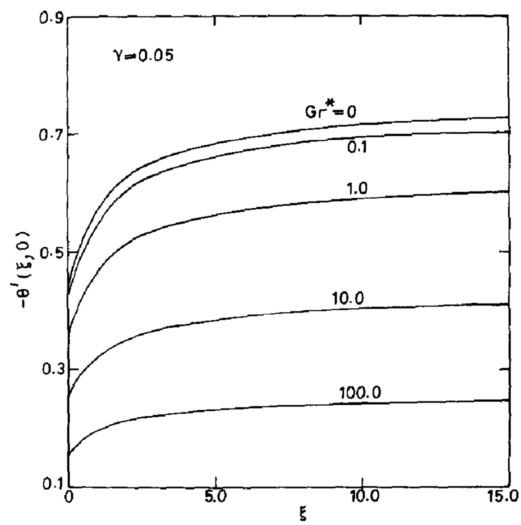

FIG. 2. Effect of the modified Grashof number Gr^* on the heat transfer parameter.

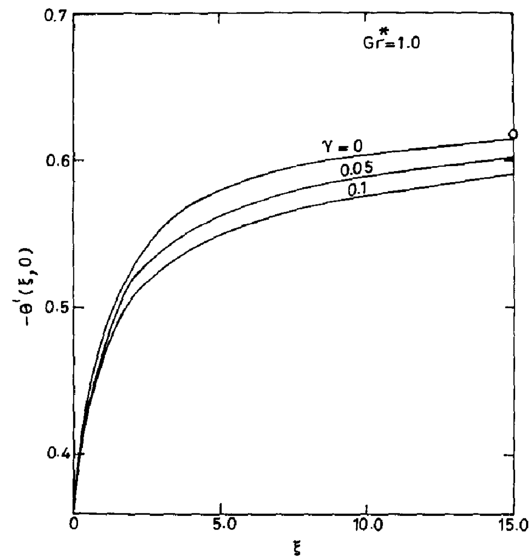

FIG. 3. Effect of the transverse curvature γ on the heat transfer parameter: —— nonsimilar solution; ○ similar solution.

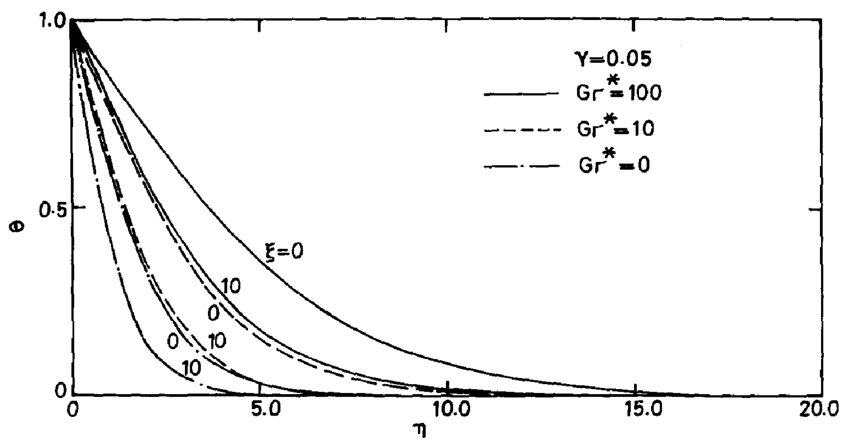

FIG. 4. Effect of the modified Grashof number Gr^* on the temperature profiles.

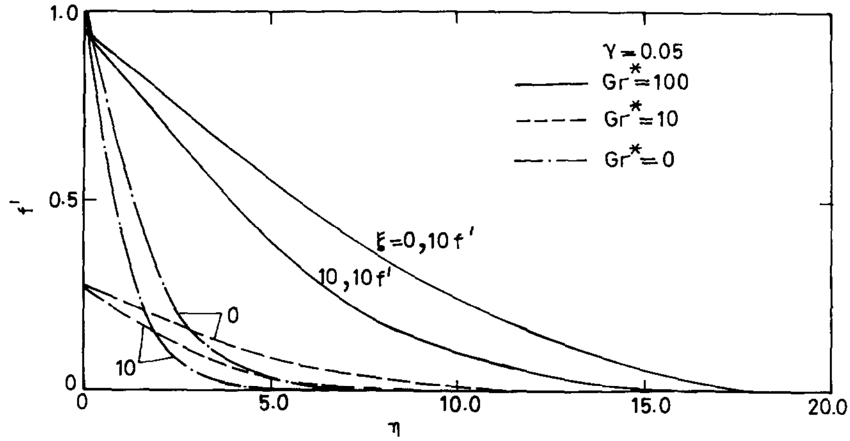

FIG. 5. Effect of the modified Grashof number Gr^* on the velocity profiles.

We have studied the effect of step sizes $\Delta\eta$ and $\Delta\xi$, and the edge of the boundary layer (η_∞) on the solution with a view to optimize them. Consequently, computations were carried out on a DEC-1090 computer with $\Delta\eta = \Delta\xi = 0.1$, $5 \leqslant \eta_\infty \leqslant 18$ depending on the values of Gr^* and ξ. The results presented here are independent of the step sizes and η_∞ at least up to the fourth decimal place. The CPU time taken by a typical data is 24.1 s.

In order to assess the accuracy of our method, we have compared our results for heat transfer and velocity at the wall [$\theta'(0)$, $f'(0)$] for $\xi = \gamma = \eta = 0$ with those of Plumb and Huenefeld [4] and found them in excellent agreement (see Table 1).

Figures 2 and 3 show the effect of the modified Grashof number Gr^* (which represents the relative importance of the inertial effects) and transverse curvature γ on the heat transfer parameter $-\theta'(\xi, 0)$, respectively. It is observed that the heat transfer $-\theta'(\xi, 0)$ decreases as Gr^* or γ increases. This behaviour is due to the increase in the thickness of the thermal boundary layer caused by the increase in Gr^* and γ. It is also observed that for a prescribed Gr^* and γ the heat transfer $-\theta'(\xi, 0)$ increases rapidly with ξ when ξ is small. For large ξ the change is very small and the heat transfer $-\theta'(\xi, 0)$ attains an asymptotic value. For $\gamma = 0$ and $\xi = 15$ the heat transfer $-\theta'(\xi, 0)$ is very nearly the same as that of the similarity solution obtained from equations (20) and (21) (see Fig. 3). The same holds good for the velocity $f'(\xi, 0)$. However, it is not shown here for the sake of brevity.

The effect of the modified Grashof number Gr^* on the temperature θ and velocity profile f' is shown in Figs. 4 and 5, respectively. It is observed that both the thermal and velocity boundary layers increase as Gr^* increases. Consequently, the temperature and velocity profiles (θ, f') become less flat as Gr^* increases. The effect of ξ is just the opposite.

CONCLUSIONS

To conclude, it might be worth mentioning that the heat transfer and velocity gradient within the convective boundary layer adjacent to a slender vertical frustum of a cone immersed in a saturated porous medium are strongly affected by the modified Grashof number (non-Darcian effect) and transverse curvature of the frustum cone. However, the effect is small for small ξ. Both the velocity and thermal boundary-layer thicknesses decrease as the modified Grashof number and transverse curvature increase.

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